HIGHER DERIVATIONS ON A SMALL PREADDITIVE CATEGORY

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Abstract. In this article, we consider a higher derivation on a preadditive category $\mathfrak{C}$. We show that the higher derivation $d = \{d^n_{x,y} : \mathfrak{C}(x,y) \longrightarrow \mathfrak{C}(x,y)\}_{(x,y)\in Ob(\mathfrak{C})^2, n\geq 0}$ on $\mathfrak{C}$ corresponds to a sequence $\{\delta^n\}_{n\geq 1}$ of derivations on $\mathfrak{C}$ such that $(n+1)d^{n+1}_{x,y} = \sum_{p=0}^{n} \delta^{p+1}_{x,y} \circ d^{n-p}_{x,y}$ for all objects $x, y \in \mathfrak{C}$ and non-negative integers n.

1. Introduction

In 1936, Hasse and Schmidt [2] considered higher derivations for the first time. Hence higher derivations are also known as *Hasse-Schmidt derivations*. Several variations of the notion of higher derivations such as finite order higher derivations, locally finite iterative higher derivations and commuting finite Schmidt higher derivations have been developed in Osborn [3], Nakai [4] and Malm [5] respectively. Higher derivations have also been studied on different algebraic structures, as in commutative and non-commutative rings [6], modules [7] and finite incidence algebras by Kay GoRodov et.al [8]. For more on higher derivations, we refer the reader to [9].

The description of higher derivations on an algebra A as compositions of ordinary derivations were independently laid out by Mirzavaziri [10], Heereema [12] and Saymeh [13]. In this paper, we use the technique of Mirzavaziri [10] to characterize higher derivations on a small preadditive category.

2. Definitions and Results

A small preadditive category was termed by MacLane [1] as *Ab-category*, i.e., a category enriched over abelian groups. In the words of Mitchell [11], a small preadditive category is similar to a ring with several objects. Unless otherwise mentioned, $\mathfrak{C}$ denotes small preadditive category. For objects x, y in $\mathfrak{C}$, we denote by $\mathfrak{C}(x,y)$ the collection of morphisms from x to y in $\mathfrak{C}$.

2.0.1. ***Definition:*** A *derivation* δ on $\mathfrak{C}$ is a collection $\delta = \{\ \delta_{x,y} : \mathfrak{C}(x,y) \longrightarrow \mathfrak{C}(x,y)\}_{(x,y)\in Ob(\mathfrak{C})^2}$ satisfying the condition

$$\delta_{x,z}(b \circ a) = \delta_{y,z}(b) \circ a + b \circ \delta_{x,y}(a)$$

for all morphisms $a \in \mathfrak{C}(x,y)$, $b \in \mathfrak{C}(y,z)$ for objects $x, y, z \in \mathfrak{C}$.

2.0.2. ***Definition:*** A sequence $d = \left\{d^n_{x,y} : \mathfrak{C}(x,y) \longrightarrow \mathfrak{C}(x,y)\right\}_{x,y\in Ob(\mathfrak{C}), n\geq 0}$ is a *higher derivation* if for every $n \geq 0$, we have:

$$d^n_{x,z}(b \circ a) = \sum_{p=0}^{n} d^p_{y,z}(b) \circ d^{n-p}_{x,y}(a)$$

for all morphisms $a \in \mathfrak{C}(x,y)$, $b \in \mathfrak{C}(y,z)$ for objects $x, y, z \in \mathfrak{C}$. It is clear from the definition that $d^0_{x,y} = id$ for any x, $y \in \mathfrak{C}$.

2.1. ***Proposition:*** *Given a higher derivation* $d = \left\{d^n_{x,z}\right\}_{(x,z)\in Ob(\mathfrak{C})^2, n\geq 0}$ *on a small preadditive category* $\mathfrak{C}$ *with* $d^0 = id$, *then there is a sequence* $\left\{\delta^n_{x,z}\right\}_{(x,z)\in Ob(\mathfrak{C})^2}$ *of derivations on* $\mathfrak{C}$ *such that*

$$(n+1)d^{n+1}_{x,z} = \sum_{p=0}^{n} \delta^{p+1}_{x,z} \circ d^{n-p}_{x,z} \tag{1}$$

for all $(x,z) \in Ob(\mathfrak{C})^2$.

Proof. We prove the proposition by induction on n. For $n = 0$, we have,

$$\begin{aligned} d^1_{x,z}(a \circ b) &= d^0_{y,z}(a) \circ d^1_{x,y}(b) + d^1_{y,z}(a) \circ d^0_{x,y}(b) \\ &= a \circ d^1_{x,y}(b) + d^1_{y,z}(a) \circ b \end{aligned}$$

The above equation indicates that, when $\delta^1 = d^1$ then δ^1 is a derivation on $\mathfrak{C}$. Let us assume that δ^p is defined and is a derivation for all $p \leq n$. Setting $\delta^{n+1} = (n+1)d^{n+1} - \sum_{p=0}^{n-1} \delta^{p+1} d^{n-p}$, we shall show that δ^{n+1} is a derivation on $\mathfrak{C}$. For composable morphisms $a, b \in \mathfrak{C}$, we have,

$$\begin{aligned} \delta^{n+1}_{x,z}(a \circ b) &= (n+1)d^{n+1}_{x,z}(a \circ b) - \sum_{p=0}^{n-1} \delta^{p+1} d^{n-p}_{x,z}(a \circ b) \\ &= (n+1) \sum_{p=0}^{n+1} d^p_{y,z}(a) \circ d^{n+1-p}_{x,y}(b) - \sum_{p=0}^{n-1} \delta^{p+1} \left(\sum_{q=0}^{n-p} d^q_{y,z}(a) - d^{n-p-q}_{x,y}(b) \right) \end{aligned}$$

Now we have,

$$\begin{aligned} \delta^{n+1}_{x,z}(a \circ b) &= \sum_{p=0}^{n+1} (n+1) d^p_{y,z}(a) \circ d^{n+1-p}_{x,y}(b) - \sum_{p=0}^{n-1} \delta^{p+1} \left(\sum_{q=0}^{n-p} d^q_{y,z}(a) \circ d^{n-p-q}_{x,y}(b) \right) \\ &= \sum_{p=0}^{n+1} (p+n+1-p) d^p_{y,z}(a) \circ d^{n+1-p}_{x,y}(b) - \sum_{p=0}^{n-1} \delta^{p+1} \left(\sum_{q=0}^{n-p} d^q_{y,z}(a) \circ d^{n-p-q}_{x,y}(b) \right) \end{aligned}$$

Since $\delta^1, \delta^2, ..., \delta^n$ are derivations,

$$\begin{aligned} \delta^{n+1}_{x,z}(a \circ b) &= \sum_{p=0}^{n+1} p d^p_{y,z}(a) \circ d^{n+1-p}_{x,y}(b) + \sum_{p=0}^{n+1} (n+1-p) d^p_{y,z}(a) \circ d^{n+1-p}_{x,y}(b) \\ &\quad - \sum_{p=0}^{n-1} \sum_{q=0}^{n-p} \left[\delta^{p+1} \left(d^q_{y,z}(a) \right) \circ d^{n-p-q}_{x,y}(b) + d^q_{y,z}(a) \circ \delta^{p+1} \left(d^{n-p-q}_{x,y}(b) \right) \right] \end{aligned}$$

Writing,

$$K = \sum_{p=0}^{n+1} p d^p_{y,z}(a) \circ d^{n+1-p}_{x,y}(b) - \sum_{p=0}^{n-1} \sum_{q=0}^{n-p} \delta^{p+1} \left(d^q_{y,z}(a) \right) \circ d^{n-p-q}_{x,y}(b)$$

$$L = \sum_{p=0}^{n+1} (n+1-p) d^p_{y,z}(a) \circ d^{n+1-p}_{x,y}(b) - \sum_{p=0}^{n-1} \sum_{q=0}^{n-p} d^q_{y,z}(a) \circ \delta^{p+1} \left(d^{n-p-q}_{x,y}(b) \right),$$

we have $\delta^{n+1}(a \circ b) = K + L$. Let us compute K and L. In the summation $\sum_{p=0}^{n-1} \sum_{q=0}^{n-p}$, we have $0 \leq p+q \leq n$ and $p \neq n$. Thus if we put $r = p+q$, then we can write it as the form $\sum_{r=0}^{n} \sum_{p+q=r, p \neq n}$. Putting $q = r - p$, we indeed have,

$$K = \sum_{p=0}^{n+1} p d^p_{y,z}(a) \circ d^{n+1-p}_{x,y}(b) - \sum_{r=0}^{n} \sum_{0 \leq p \leq r, p \neq n} \delta^{p+1} \left(d^{r-p}_{y,z}(a) \right) \circ d^{n-r}_{x,y}(b)$$

$$= \sum_{p=0}^{n+1} p d^p_{y,z}(a) \circ d^{n+1-p}_{x,y}(b) - \sum_{r=0}^{n-1} \sum_{p=0}^{r} \delta^{p+1} \left(d^{r-p}_{y,z}(a) \right) \circ d^{n-r}_{x,y}(b) - \sum_{p=0}^{n-1} \delta^{p+1} \left(d^{n-p}_{y,z}(a) \right) \circ b.$$

Putting $r+1$ instead of p in the first summation, we have

$$K+\sum_{p=0}^{n-1}\delta^{p+1}\left(d_{y,z}^{n-p}(a)\right)\circ b$$

$$=\sum_{r=0}^{n}(r+1)d_{y,z}^{r+1}(a)\circ d_{x,y}^{n-r}(b)-\sum_{r=0}^{n-1}\sum_{p=0}^{r}\delta^{p+1}\left(d_{y,z}^{r-p}(a)\right)\circ d_{x,y}^{n-r}(b)$$

$$=\sum_{r=0}^{n-1}\left[(r+1)d_{y,z}^{r+1}(a)-\sum_{p=0}^{r}\delta^{p+1}(d_{y,z}^{r-p}(a))\right]\circ d_{x,y}^{n-r}(b)+(n+1)d_{y,z}^{n+1}(a)\circ b \quad .$$

By our assumption $(r+1)d_{y,z}^{r+1}(a)=\sum_{p=0}^{r}\delta^{p+1}(d_{y,z}^{r-p}(a))$ for $r=0,.....n-1$. We can therefore deduce that,

$$K=\left[(n+1)d_{y,z}^{n+1}(a)-\sum_{p=0}^{n-1}\delta^{p+1}\left(d_{y,z}^{n-p}(a)\right)\right]\circ b=\delta_{y,z}^{n+1}(a)\circ b.$$

By a similar argument, we have

$$L=a\circ\left[(n+1)d_{x,y}^{n+1}(b)-\sum_{p=0}^{n-1}\delta^{p+1}\left(d_{x,y}^{n-p}(b)\right)\right]=a\circ\delta_{x,y}^{n+1}(b).$$

Thus,

$$\delta_{x,z}^{n+1}(a\circ b)=K+L=\delta_{y,z}^{n+1}(a)\circ b+a\circ\delta_{x,y}^{n+1}(b)$$

It follows that δ^{n+1} is a derivation on $\mathfrak{C}$. □

2.2. **Theorem:** *In a small preadditive category $\mathfrak{C}$ enriched over $\mathbb{Q}$-vector spaces, there is a one-one correspondence between higher derivations $d=\left\{d_{x,y}^{n}:\mathfrak{C}(x,y)\longrightarrow\mathfrak{C}(x,y)\right\}_{(x,y)\in Ob(\mathfrak{C})^2,n\geq 0}$ on $\mathfrak{C}$ and sequences of derivations $\left\{\delta^{n}\right\}_{n\geq 1}$ on $\mathfrak{C}$.*

Proof. Since $\mathfrak{C}$ is a $\mathbb{Q}$-linear category, it is clear from the expression in (1) that the correspondence between higher derivations on $\mathfrak{C}$ and sequences of ordinary derivations on $\mathfrak{C}$ is a bijection. □

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